

# Single Beacon Navigation: Observability Analysis and Filter Design

Pedro Batista, Carlos Silvestre, and Paulo Oliveira

**Abstract**—This paper addresses the problem of navigation of autonomous vehicles based on the range to a single beacon. The vehicle is equipped with a standard Inertial Measurement Unit (IMU) and range measurements to a single source are available as an aiding observation, in addition to angular velocity readings. The contribution of the paper is twofold: i) necessary and sufficient conditions on the observability of the system are derived that can be used for the motion planning and control of the vehicle; ii) a linear model is developed that mimics the exact dynamics of the nonlinear range-based system, and a Kalman filter is applied to estimate the relative position of the source, as well as the linear velocity of the vehicle and the acceleration of gravity, all expressed in body-fixed coordinates. Simulation results are presented in the presence of realistic measurement noise that illustrate the performance achieved with the proposed solution.

## I. INTRODUCTION

Accurate navigation systems are essential for the successful operation of autonomous vehicles. Although there exist alternatives such as terrain-based navigation, most navigation systems contain an Inertial Navigation System (INS) that provides the state of the vehicle by integrating, in open-loop, the information provided by inertial sensors, e.g., accelerometers and rate gyros. Although INS provides very good short term results, its performance necessarily degrades over time, which has lead the scientific community to consider aiding devices in the design of Navigation Systems. Among the myriad of aiding devices, the Global Positioning System (GPS) is a very popular choice, see, e.g., [1], [2], and [3]. While long baseline (LBL) solutions offer more information, recent efforts have been made in the field of navigation using range measurements to a single source. This last solution, although harder from the theoretical point of view, has significant cost advantages. Moreover, the complexity of the hardware is lower and, for intensive missions, the deployment time is also much smaller. A so-called Synthetic Long Baseline (SLBL) navigation algorithm for underwater vehicles is proposed in [4]. The vehicle is assumed to have access to range measurements to a single transponder, from time to time, and between sampling instants, a high performance dead-reckoning system is used to extrapolate the motion of the vehicle. A discrete-time Kalman filter is applied to a linearized model of the system to obtain the required estimates. In [5] the authors deal with the problem of underwater navigation in the presence of unknown currents based on range measurements to a single

This work was partially funded by Fundação para a Ciência e a Tecnologia (ISR/IST plurianual funding) through the PIIDAC Program funds and EU Project TRIDENT (Contract No. 248497).

The work of P. Batista was supported by a PhD Student Scholarship from the POCTI Programme of FCT, SFRH/BD/24862/2005.

The authors are with the Institute for Systems and Robotics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. [{pbatista,cjs,pjcro}@isr.ist.utl.pt](mailto:{pbatista,cjs,pjcro}@isr.ist.utl.pt)

beacon. An observability analysis is presented based on the linearization of the nonlinear system which yields, off course, local results. Based on the linearized system dynamics, a Luenberger observer is introduced but in practice an extended Kalman filter (EKF) is implemented, with no warranties of global asymptotic stability. More recently, the same problem has been studied in [6] and [7], where EKFs have been extensively used to solve the navigation problem based on single beacon range measurements.

This paper addresses the problem of vehicle navigation based on range measurements to a single source. The contribution is twofold: i) the observability of the nonlinear system is analyzed and necessary and sufficient conditions derived that characterize this aspect of the system; ii) a filter design methodology is proposed based on a linear model for the system that captures the exact dynamics of the nonlinear system. Central to the observability analysis and the filter design is the derivation of a linear time-varying (LTV) system that captures the dynamics of the nonlinear system. The LTV model is achieved through appropriate state augmentation, which is shown to mimic the nonlinear system under specific observability conditions. Applications of the proposed solution are many and an example is shown with a quadrotor, although the theoretical setting is completely general and can be applied to underwater, surface, aerial or space vehicles.

The paper is organized as follows. The problem statement and system dynamics are introduced in Section II. Section III refers to the observability analysis, while the filter design is proposed in Section IV. Simulation results are presented in Section V and Section VI summarizes the main conclusions of the paper.

### A. Notation

Throughout the paper the symbol  $\mathbf{0}$  denotes a matrix (or vector) of zeros and  $\mathbf{I}$  an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as  $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ . If  $\mathbf{x}$  and  $\mathbf{y}$  are two vectors of identical dimensions,  $\mathbf{x} \times \mathbf{y}$  and  $\mathbf{x} \cdot \mathbf{y}$  represent the cross and inner products, respectively.

## II. PROBLEM STATEMENT

### A. Motivation

Consider an autonomous vehicle moving in a scenario where a fixed source is installed and suppose that the vehicle measures the range to the source. In the field of ocean robotics, e.g., the range to an acoustic beacon may be determined from the time-of-flight of an acoustic signal, whereas in space applications electromagnetic pulses may be employed. The problem considered in the paper is that of estimating the position of the source relative to the

vehicle based on the range measurements. If the position of the source is known in inertial coordinates then, by estimating the position of the source relative to the vehicle, the vehicle can, in fact, navigate in inertial coordinates. Other applications may require only the position of the source relative to the vehicle, e.g., homing and docking tasks [8].

Knowledge of the range to the source is, evidently, insufficient to determine its position. Indeed, the vehicle must have some information about its own motion and, moreover, not all trajectories render the position of the source observable, as it will be shown in the sequel. To complete the navigation sensor suite of the vehicle, the vehicle is assumed to be equipped with an Inertial Measurement Unit (IMU), containing two triads of orthogonally mounted accelerometers and rate-gyros that provide the linear acceleration and the angular velocity of the vehicle, respectively.

To properly set the problem framework, let  $\{I\}$  denote a local inertial reference coordinate frame and  $\{B\}$  a coordinate frame attached to the vehicle, denominated in the sequel as the body-fixed coordinate frame. The linear motion of the vehicle is described by  $\dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t)$ , where  $\mathbf{p} \in \mathbb{R}^3$  denotes the inertial position of the vehicle,  $\mathbf{v} \in \mathbb{R}^3$  the velocity of the vehicle relative to  $\{I\}$  and expressed in body-fixed coordinates, and  $\mathbf{R} \in SO(3)$  is the rotation matrix from  $\{B\}$  to  $\{I\}$ , which satisfies  $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}(\omega(t))$ , where  $\omega \in \mathbb{R}^3$  is the angular velocity of  $\{B\}$ , expressed in body-fixed coordinates, and  $\mathbf{S}(\omega)$  is the skew-symmetric matrix such that  $\mathbf{S}(\omega)\mathbf{x}$  is the cross product  $\omega \times \mathbf{x}$ . Let  $\mathbf{s}$  denote the inertial position of the source. Then, the range to the source is given by  $r(t) = \|\mathbf{r}(t)\|$ , where

$$\mathbf{r}(t) := \mathbf{R}^T(t)[\mathbf{s} - \mathbf{p}(t)] \quad (1)$$

is the location of the source relative to the vehicle, expressed in body-fixed coordinates, precisely the quantity that the vehicle aims to estimate. The rotation matrix  $\mathbf{R}$  is used in this paper for theoretical purposes only and is not needed in practice. Should it be needed, an Attitude and Heading Reference System would provide this quantity. The angular velocity  $\omega$  is provided by the IMU. Finally, the triad of accelerometers reads

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) + \mathbf{S}(\omega(t))\mathbf{v}(t) - \mathbf{g}(t), \quad (2)$$

where  $\mathbf{g}(t) \in \mathbb{R}^3$  denotes the acceleration of gravity, expressed in body-fixed coordinates. Ideal accelerometers would not measure the gravitational term but in practice this term must be considered due to the inherent physics of the accelerometers, see [9] for further details. The term  $\mathbf{S}(\omega(t))\mathbf{v}(t)$  corresponds to the Coriolis acceleration and cannot be neglected, particularly for vehicles that execute aggressive maneuvers. The vehicle aims to estimate the source localization  $\mathbf{r}(t)$ , as well as the vehicle linear velocity  $\mathbf{v}(t)$ , from the knowledge of  $r(t)$ ,  $\mathbf{a}(t)$ ,  $\mathbf{R}(t)$ , and  $\omega(t)$ .

### B. System Dynamics

The time derivative of (1) is given by

$$\dot{\mathbf{r}}(t) = -\mathbf{v}(t) - \mathbf{S}(\omega(t))\mathbf{r}(t)$$

and, from (2), it follows that

$$\dot{\mathbf{v}}(t) = \mathbf{a}(t) - \mathbf{S}(\omega(t))\mathbf{v}(t) + \mathbf{g}(t).$$

Although the acceleration of gravity may be known with great accuracy in inertial coordinates, any misalignment in this quantity, when expressed in body-fixed coordinates, could lead to severe accuracy degradation. To avoid that,  $\mathbf{g}(t)$  is also considered as a state in this framework. Assuming that the acceleration of gravity is constant in inertial coordinates then it follows that  $\dot{\mathbf{g}}(t) = -\mathbf{S}(\omega(t))\mathbf{g}(t)$ . The final system dynamics can be written as

$$\begin{cases} \dot{\mathbf{r}}(t) = -\mathbf{v}(t) - \mathbf{S}(\omega(t))\mathbf{r}(t) \\ \dot{\mathbf{v}}(t) = \mathbf{a}(t) - \mathbf{S}(\omega(t))\mathbf{v}(t) + \mathbf{g}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}(\omega(t))\mathbf{g}(t) \\ y(t) = \|\mathbf{r}(t)\| \end{cases}. \quad (3)$$

The problem considered in the paper is the analysis of the observability of the nonlinear time-varying system (3) and the design of a state observer to estimate the system states.

### III. OBSERVABILITY ANALYSIS

While the observability of linear systems is nowadays fairly well understood, the observability of nonlinear systems is still an open field of research, as evidenced (and in spite of) the large number of recent publication on the subject. The analysis of the linearization of a nonlinear system does not yield definite results either. This section provides an analysis of the observability of the nonlinear time-varying system (3) by means of state augmentation.

#### A. Coordinate Transformation

Let  $\mathbf{T}(t) := \text{diag}(\mathbf{R}(t), \mathbf{R}(t), \mathbf{R}(t))$  and consider the state transformation

$$\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \mathbf{x}_3(t) \end{bmatrix} := \mathbf{T}(t) \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \\ \mathbf{g}(t) \end{bmatrix}, \quad (4)$$

which is a Lyapunov state transformation previously used by the authors [10]. The new system dynamics are given by

$$\begin{cases} \dot{\mathbf{x}}_1(t) = -\mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{x}_3(t) + \mathbf{u}(t) \\ \dot{\mathbf{x}}_3(t) = \mathbf{0} \\ y(t) = \|\mathbf{x}_1(t)\| \end{cases}, \quad (5)$$

where  $\mathbf{u}(t) := \mathbf{R}(t)\mathbf{a}(t)$ . Notice that as (4) is a Lyapunov state transformation all observability properties are kept [11].

#### B. State Augmentation

To derive a linear system that mimics the dynamics of the nonlinear system (5), define five additional scalar state variables as

$$\begin{cases} x_4(t) := y(t) \\ x_5(t) := \mathbf{x}_1(t) \cdot \mathbf{x}_2(t) \\ x_6(t) := \mathbf{x}_1(t) \cdot \mathbf{x}_3(t) - \|\mathbf{x}_2(t)\|^2 \\ x_7(t) := \mathbf{x}_2(t) \cdot \mathbf{x}_3(t) \\ x_8(t) := \|\mathbf{x}_3(t)\|^2 \end{cases}$$

and denote by

$$\mathbf{x}(t) = [\mathbf{x}_1^T(t) \ \mathbf{x}_2^T(t) \ \mathbf{x}_3^T(t) \ x_4(t) \ \dots \ x_8(t)]^T \in \mathbb{R}^n,$$

$n = 9 + 5$ , the augmented state. It is easy to verify that the dynamics of the augmented system can be written as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases}, \quad (6)$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{y(t)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{u}^T(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -2\mathbf{u}^T(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{u}^T(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (7)$$

$\mathbf{B} = [\mathbf{0} \ \mathbf{I} \ \mathbf{0} \ \mathbf{0} \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}]^T$ , and  $\mathbf{C} = [\mathbf{0} \ \mathbf{0} \ \mathbf{0} \ 1 \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}]$ .

The dynamic system (6) can be regarded as a linear time-varying system, even though the system matrix  $\mathbf{A}(t)$  depends explicitly on the system input and output, as evidenced by (7). Nevertheless, this is not a problem from the theoretical point of view since both the input and output of the system are known continuous bounded signals. The idea is not new either, see, e.g., [12], and it just suggests, in this case, that the observability of (6) may be connected with the evolution of the system input or output (or both), which is not common and does not happen when this matrix does not depend on the system input or output. The observability analysis of (6) will follow using classical theory of linear systems. Notice that there is nothing in the system (6) imposing

$$\begin{cases} y(t) = \|\mathbf{x}_1(t)\| \\ x_5(t) = \mathbf{x}_1(t) \cdot \mathbf{x}_2(t) \\ x_6(t) = \mathbf{x}_1(t) \cdot \mathbf{x}_3(t) - \|\mathbf{x}_2(t)\|^2 \\ x_7(t) = \mathbf{x}_2(t) \cdot \mathbf{x}_3(t) \\ x_8(t) = \|\mathbf{x}_3(t)\|^2 \end{cases}. \quad (8)$$

Although these restrictions could be easily imposed including artificial outputs, e.g.,  $y_2(t) = x_5(t) - \mathbf{x}_1(t) \cdot \mathbf{x}_2(t) = 0 \ \forall t$ , this form is preferred since it allows to consider the system as linear. However, care must be taken when extrapolating conclusions from the observability of (6) to the observability of (5) or (3). Finally, notice that (7) is only well defined for  $y(t) \neq 0$ . This is a mild assumption as, for  $y(t) = 0$ , the location of the vehicle coincides with the source location, which is impossible in practice. Therefore, the following assumption is introduced.

*Assumption 1:* The motion of the vehicle is such that

$$\begin{array}{lll} \exists & \forall & : Y_m \leq y(t) \leq Y_M. \\ Y_m > 0 & t > t_0 \\ Y_M > 0 \end{array}$$

*Remark 1:* It is possible to derive all the results in the paper without Assumption 1 if a different augmented state vector is considered, using the square of the range instead of the range itself. Although this has no disadvantages from the theoretical point of view, it presents, in practice, a significant inconvenient as sensor noise is greatly amplified by the square operation for large transponder-vehicle distances. Therefore, the use of the range instead of its square is preferred in the paper.

### C. Observability of the LTV System

In order to proceed with the analysis of the observability of the LTV system (6), it is convenient to compute the observability Gramian associated with the pair  $(\mathbf{A}(t), \mathbf{C})$  and, in order to do so, the transition matrix associated with  $\mathbf{A}(t)$ . Let  $\mathbf{u}^{[1]}(t, t_0) := \int_{t_0}^t \mathbf{u}(\sigma) d\sigma$  and

$$\mathbf{u}^{[2]}(t, t_0) := \int_{t_0}^t \int_{t_0}^{\sigma_1} \mathbf{u}(\sigma_2) d\sigma_2 d\sigma_1 = \begin{bmatrix} u_1^{[2]}(t, t_0) \\ u_2^{[2]}(t, t_0) \\ u_3^{[2]}(t, t_0) \end{bmatrix},$$

where  $(.)^{[i]}$  represents the  $i^{th}$  integral quantity. Then, it is straightforward to show that the transition matrix associated with  $\mathbf{A}(t)$  is given by

$$\phi(t, t_0) = \begin{bmatrix} \phi_A(t, t_0) & | & \mathbf{0} \\ \hline \phi_4^T(t, t_0) & | & \phi_B(t, t_0) \\ \hline \phi_C(t, t_0) & | & \phi_C(t, t_0) \end{bmatrix},$$

with

$$\phi_A(t, t_0) = \begin{bmatrix} \mathbf{I} & -(t-t_0)\mathbf{I} & -\frac{(t-t_0)^2}{2}\mathbf{I} \\ \mathbf{0} & \mathbf{I} & (t-t_0)\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\phi_4(t, t_0) = \begin{bmatrix} -\int_{t_0}^t \frac{\mathbf{u}^{[1]}(\sigma, t_0) d\sigma}{y(\sigma)} \\ \int_{t_0}^t \frac{(\sigma-t_0)\mathbf{u}^{[1]}(\sigma, t_0) + \mathbf{u}^{[2]}(\sigma, t_0) d\sigma}{y(\sigma)} \\ \int_{t_0}^t \frac{(\sigma-t_0)^2/2\mathbf{u}^{[1]}(\sigma, t_0) + (\sigma-t_0)\mathbf{u}^{[2]}(\sigma, t_0) d\sigma}{y(\sigma)} \\ 1 \\ -\int_{t_0}^t \frac{1}{y(\sigma)} d\sigma \\ -\int_{t_0}^t \frac{\sigma-t_0}{y(\sigma)} d\sigma \\ \int_{t_0}^t \frac{3(\sigma-t_0)^2}{2y(\sigma)} d\sigma \\ \int_{t_0}^t \frac{1}{2} \frac{(\sigma-t_0)^3}{y(\sigma)} d\sigma \end{bmatrix},$$

and  $\phi_B(t, t_0)$  and  $\phi_C(t, t_0)$  are omitted as they are not required in the sequel. The observability Gramian associated with the pair  $(\mathbf{A}(t), \mathbf{C})$  is simply given by

$$\mathcal{W}(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(t, t_0) \mathbf{C}^T \mathbf{C} \phi(t, t_0) dt.$$

The following theorem establishes a necessary and sufficient condition on the observability of the LTV system (6).

*Theorem 1:* The LTV system (6) is observable on  $[t_0, t_f]$ ,  $t_0 < t_f$ , if and only if the set of functions

$$\begin{aligned} \mathcal{F} = & \left\{ t - t_0, (t - t_0)^2, (t - t_0)^3, (t - t_0)^4 \right. \\ & u_1^{[2]}(t, t_0), u_2^{[2]}(t, t_0), u_3^{[2]}(t, t_0), \\ & (t - t_0)u_1^{[2]}(t, t_0), (t - t_0)u_2^{[2]}(t, t_0), (t - t_0)u_3^{[2]}(t, t_0), \\ & \left. (t - t_0)^2u_1^{[2]}(t, t_0), (t - t_0)^2u_2^{[2]}(t, t_0), (t - t_0)^2u_3^{[2]}(t, t_0) \right\} \end{aligned}$$

is linearly independent on  $[t_0, t_f]$ .

*Proof:*

a) *Sufficiency:* Suppose that the LTVS (6) is not observable on  $[t_0, t_f]$ . Then, the observability Gramian  $\mathcal{W}(t_0, t_f)$  is not positive definite and therefore

$$\begin{array}{lll} \exists & \forall & : \mathbf{d}^T \mathcal{W}(t_0, t) \mathbf{d} = 0. \\ \mathbf{d} \in \mathbb{R}^n & t \in [t_0, t_f] \\ \|\mathbf{d}\| = 1 \end{array} \quad (9)$$

Expanding (9) gives  $\int_{t_0}^t \|\phi_4(\sigma, t_0) \cdot \mathbf{d}\|^2 d\sigma = 0$  for all  $t \in [t_0, t_f]$ , which implies that

$$\phi_4(t, t_0) \cdot \mathbf{d} = 0 \quad (10)$$

for all  $t \in [t_0, t_f]$ . Let  $\mathbf{d} = [\mathbf{d}_1^T \mathbf{d}_2^T \mathbf{d}_3^T d_4 \dots d_8]^T$ . Notice that  $\phi_4(t_0, t_0) \cdot \mathbf{d} = d_4$  and therefore, for (10) to hold, it must be  $d_4 = 0$ . On the other hand, from (10) it also follows that  $\frac{d}{dt} \phi_4(t, t_0) \cdot \mathbf{d} = 0$  for all  $t \in [t_0, t_f]$ , which implies that

$$\begin{aligned} -\mathbf{u}^{[1]}(t, t_0) \cdot \mathbf{d}_1 + [(t-t_0) \mathbf{u}^{[1]}(t, t_0) + \mathbf{u}^{[2]}(t, t_0)] \cdot \mathbf{d}_2 \\ \left[ \frac{(t-t_0)^2}{2} \mathbf{u}^{[1]}(t, t_0) + (t-t_0) \mathbf{u}^{[2]}(t, t_0) \right] \cdot \mathbf{d}_3 \\ -d_5 - (t-t_0) d_6 + \frac{3}{2} (t-t_0)^2 d_7 \\ + \frac{1}{2} (t-t_0)^3 d_8 = 0 \end{aligned} \quad (11)$$

for all  $t \in [t_0, t_f]$ . Integrating both sides of (11) gives

$$\begin{aligned} -\mathbf{u}^{[2]}(t, t_0) \cdot \mathbf{d}_1 + (t-t_0) \mathbf{u}^{[2]} \cdot \mathbf{d}_2 \\ + \frac{(t-t_0)^2}{2} \mathbf{u}^{[2]}(t, t_0) \cdot \mathbf{d}_3 - (t-t_0) d_5 - \frac{(t-t_0)^2}{2} d_6 \\ + \frac{(t-t_0)^3}{2} d_7 + \frac{(t-t_0)^4}{8} d_8 = 0, \end{aligned}$$

which means that the set of functions  $\mathcal{F}$  is not linearly independent on  $[t_0, t_f]$ . Therefore, if the set of functions  $\mathcal{F}$  is linearly independent on  $[t_0, t_f]$ , (6) is observable on  $[t_0, t_f]$ .

b) *Necessity:* Suppose that the set of functions  $\mathcal{F}$  is not linearly independent on  $[t_0, t_f]$ . Then, there exists a nonzero vector  $\mathbf{d} = [\mathbf{d}_1^T \mathbf{d}_2^T \mathbf{d}_3^T d_4 d_5 d_6 d_7]^T$  such that

$$\begin{aligned} -\mathbf{u}^{[2]}(t, t_0) \cdot \mathbf{d}_1 + (t-t_0) \mathbf{u}^{[2]} \cdot \mathbf{d}_2 \\ + \frac{(t-t_0)^2}{2} \mathbf{u}^{[2]}(t, t_0) \cdot \mathbf{d}_3 - (t-t_0) d_5 - \frac{(t-t_0)^2}{2} d_6 \\ + \frac{(t-t_0)^3}{2} d_7 + \frac{(t-t_0)^4}{8} d_8 = 0 \end{aligned} \quad (12)$$

for all  $t \in [t_0, t_f]$ , and taking the time derivative of (12),

$$\begin{aligned} -\mathbf{u}^{[1]}(t, t_0) \cdot \mathbf{d}_1 + [(t-t_0) \mathbf{u}^{[1]}(t, t_0) + \mathbf{u}^{[2]}(t, t_0)] \cdot \mathbf{d}_2 \\ \left[ \frac{(t-t_0)^2}{2} \mathbf{u}^{[1]}(t, t_0) + (t-t_0) \mathbf{u}^{[2]}(t, t_0) \right] \cdot \mathbf{d}_3 \\ -d_5 - (t-t_0) d_6 + \frac{3}{2} (t-t_0)^2 d_7 \\ + \frac{1}{2} (t-t_0)^3 d_8 = 0. \end{aligned}$$

Selecting  $\mathbf{d}^* = [\mathbf{d}_1^T \mathbf{d}_2^T \mathbf{d}_3^T 0 d_4 d_5 d_6 d_7]^T$ , it is clear that  $\mathbf{d}^{*T} \mathbf{W}(t_0, t) \mathbf{d}^* = 0$  for all  $t \in [t_0, t_f]$ , which means that (6) is not observable on  $[t_0, t_f]$ . Therefore, if (6) is observable on  $[t_0, t_f]$ , the set of functions  $\mathcal{F}$  is linearly independent on  $[t_0, t_f]$ , which concludes the proof. ■

The linear independence condition on the set of functions  $\mathcal{F}$  provides little insight on the motion required by the vehicle so that observability is attained. This is established in the following theorem.

**Theorem 2:** The LTV system (6) is observable on  $[t_0, t_f]$ ,  $t_0 < t_f$ , if and only if the set of functions

$$\begin{aligned} \mathcal{F}^* = & \{t-t_0, (t-t_0)^2, (t-t_0)^3, (t-t_0)^4, \\ & p_1(t) - p_1(t_0), p_2(t) - p_2(t_0), p_3(t) - p_3(t_0), \\ & (t-t_0)[p_1(t) - p_1(t_0)], (t-t_0)[p_2(t) - p_2(t_0)], \\ & (t-t_0)[p_3(t) - p_3(t_0)], (t-t_0)^2[p_1(t) - p_1(t_0)], \\ & (t-t_0)^2[p_2(t) - p_2(t_0)], (t-t_0)^2[p_3(t) - p_3(t_0)]\} \end{aligned}$$

is linearly independent on the interval  $[t_0, t_f]$ , where  $\mathbf{p}(t) = [p_1(t) \ p_2(t) \ p_3(t)]^T$ .

*Proof:* As it has been shown, in Theorem 1, that the LTV system (6) is observable on  $[t_0, t_f]$  if and only if the set of functions  $\mathcal{F}$  is linearly independent on that interval, the proof of the theorem follows by establishing that the set of functions  $\mathcal{F}$  is linearly independent on  $[t_0, t_f]$  if and only if so is the set of functions  $\mathcal{F}^*$ . Expressing  $\mathbf{u}^{[2]}(t, t_0)$  as a function of the motion of the vehicle and the initial velocity and gravity, in inertial coordinates, gives

$$\mathbf{u}^{[2]}(t, t_0) = \mathbf{p}(t) - \mathbf{p}(t_0) - (t-t_0)\mathbf{x}_2(t_0) - \frac{(t-t_0)^2}{2}\mathbf{x}_3(t_0). \quad (13)$$

Substituting (13) in the set of functions  $\mathcal{F}$ , it becomes obvious that the set of functions  $\mathcal{F}$  is linearly independent on  $[t_0, t_f]$  if and only if so is the set of functions  $\mathcal{F}^*$ , which concludes the proof. ■

#### D. Observability of the Nonlinear System

The observability of the nonlinear system (3) is discussed in this section. The following theorem provides a sufficient observability condition and a practical result on the design of state observers for the nonlinear system (5).

**Theorem 3:** Suppose that the set of functions  $\mathcal{F}^*$  is linearly independent on  $[t_0, t_f]$ . Then,

- 1) the nonlinear system (5) is observable on  $[t_0, t_f]$ ;
- 2) a state observer with globally asymptotically stable error dynamics for the LTV system (6) is also a state observer for the nonlinear system (5), with globally asymptotically stable error dynamics.

*Proof:* Let  $[\mathbf{x}_1^T(t_0) \ \mathbf{x}_2^T(t_0) \ \mathbf{x}_3^T(t_0)]^T$  be the initial state of the nonlinear system (5). Then, the output at time  $t$  is given by

$$y(t) = \sqrt{\|\mathbf{x}_1(t)\|^2} \quad (14)$$

where

$$\begin{aligned} \|\mathbf{x}_1(t)\|^2 &= \left\| \mathbf{x}_1(t_0) - (t-t_0)\mathbf{x}_2(t_0) - \frac{(t-t_0)^2}{2}\mathbf{x}_3(t_0) - \mathbf{u}^{[2]}(t, t_0) \right\|^2 \\ &= \|\mathbf{x}_1(t_0)\|^2 + \|\mathbf{x}_2(t_0)\|^2 (t-t_0)^2 + \frac{(t-t_0)^4}{4} \|\mathbf{x}_3(t_0)\|^2 \\ &\quad - 2(t-t_0)\mathbf{x}_1(t_0) \cdot \mathbf{x}_2(t_0) - (t-t_0)^2 \mathbf{x}_1(t_0) \cdot \mathbf{x}_3(t_0) \\ &\quad + (t-t_0)^3 \mathbf{x}_2(t_0) \cdot \mathbf{x}_3(t_0) + \|\mathbf{u}^{[2]}(t, t_0)\|^2 \\ &\quad - 2\mathbf{x}_1(t_0) \cdot \mathbf{u}^{[2]}(t, t_0) + 2(t-t_0)\mathbf{x}_2(t_0) \cdot \mathbf{u}^{[2]}(t, t_0) \\ &\quad + (t-t_0)^2 \mathbf{x}_3(t_0) \cdot \mathbf{u}^{[2]}(t, t_0). \end{aligned}$$

Assuming that the set of functions  $\mathcal{F}^*$  is linearly independent on  $[t_0, t_f]$ , then it follows, from Theorem 2, that the LTVS (6) is observable on  $[t_0, t_f]$ . Thus, given  $\{y(t) : t \in [t_0, t_f]\}$ , the initial state of (6) is uniquely determined. Let

$$[\mathbf{z}_1^T(t_0) \ \mathbf{z}_2^T(t_0) \ \mathbf{z}_3^T(t_0) \ z_4(t_0) \ z_5(t_0) \ z_6(t_0) \ z_7(t_0) \ z_8(t_0)]^T$$

be the initial state of the linear system (6). Then, the square of the output satisfies

$$\begin{aligned} y^2(t) &= -2\mathbf{z}_1(t_0) \cdot \mathbf{u}^{[2]}(t, t_0) + 2(t-t_0)\mathbf{z}_2(t_0) \cdot \mathbf{u}^{[2]}(t, t_0) \\ &\quad + (t-t_0)^2 \mathbf{z}_3(t_0) \cdot \mathbf{u}^{[2]}(t, t_0) + z_4(t_0) - 2(t-t_0)z_5(t_0) \\ &\quad - (t-t_0)^2 z_6(t_0) + (t-t_0)^3 z_7(t_0) \\ &\quad + \frac{(t-t_0)^4}{4} z_8(t_0) + \|\mathbf{u}^{[2]}(t, t_0)\|^2. \end{aligned} \quad (15)$$

From the comparison between (14) and (15) it follows that

$$\begin{aligned}
& -2[\mathbf{x}_1(t_0) - \mathbf{z}_1(t_0)] \cdot \mathbf{u}^{[2]}(t, t_0) \\
& + 2[\mathbf{x}_2(t_0) - \mathbf{z}_2(t_0)] \cdot (t - t_0) \mathbf{u}^{[2]}(t, t_0) \\
& + 2[\mathbf{x}_3(t_0) - \mathbf{z}_3(t_0)] \cdot (t - t_0)^2 \mathbf{u}^{[2]}(t, t_0) \\
& + [\|\mathbf{x}_1(t_0)\| - z_4(t_0)] \\
& - 2[\mathbf{x}_1(t_0) \cdot \mathbf{x}_2(t_0) - z_5(t_0)](t - t_0) \\
& - [\mathbf{x}_1(t_0) \cdot \mathbf{x}_3(t_0) - \|\mathbf{x}_2(t_0)\|^2 - z_6(t_0)](t - t_0)^2 \\
& + [\mathbf{x}_2(t_0) \cdot \mathbf{x}_3(t_0) - z_7(t_0)](t - t_0)^3 \\
& + [\|\mathbf{x}_3(t_0)\|^2 - z_8(t_0)] \frac{(t-t_0)^4}{4} = 0. \quad (16)
\end{aligned}$$

for all  $t \in [t_0, t_f]$ . Notice that  $z_4(t_0) = \|\mathbf{x}_1(t_0)\|$ . Since it is assumed that the set of functions  $\mathcal{F}^*$  is linearly independent, it is easy to see that the only solution of (16) is  $\mathbf{x}_1(t_0) = \mathbf{z}_1(t_0)$ ,  $\mathbf{x}_2(t_0) = \mathbf{z}_2(t_0)$ ,  $\mathbf{x}_3(t_0) = \mathbf{z}_3(t_0)$ ,  $\|\mathbf{x}_1(t_0)\|^2 = z_4(t_0)$ ,  $\mathbf{x}_1(t_0) \cdot \mathbf{x}_2(t_0) = z_5(t_0)$ ,  $\mathbf{x}_1(t_0) \cdot \mathbf{x}_3(t_0) - \|\mathbf{x}_2(t_0)\|^2 = z_6(t_0)$ ,  $\mathbf{x}_2(t_0) \cdot \mathbf{x}_3(t_0) - z_7(t_0)$ , and  $\|\mathbf{x}_3(t_0)\|^2 = z_8(t_0)$ . This concludes the proof, as both the initial state of the nonlinear system (5) is uniquely determined and the initial state of the linear (6) system matches the initial state of the nonlinear system, in spite of the fact that the algebraic restrictions (8) were not explicitly imposed. ■

Notice that the definition of observability for nonlinear systems does not imply that every admissible input distinguishes points of the state space, although that is true for linear systems [13]. Nevertheless, that is implied in Theorem 3 since the core system for the proof of the result is linear.

Theorem 3 introduces a sufficient observability condition for the nonlinear system (5) that requires the linear independence of 13 functions, which may be considered, at first glance, a conservative result attending to the fact that the original nonlinear system has only 9 states. If on one hand this is due to the fact that the original system dynamics (5) were augmented but the algebraic restrictions (8) were not explicitly imposed, on the other hand it will be shown in the sequel that this linear independence condition is not that conservative. The following proposition establishes a lower bound on the number and nature of functions necessarily required to be linear independent.

*Proposition 1:* If the nonlinear system (5) is observable on  $[t_0, t_f]$ ,  $t_0 < t_f$ , then the set of functions

$$\begin{aligned}
\mathcal{F}_r = & \left\{ \mathbf{u}_1^{[2]}(t, t_0), \mathbf{u}_2^{[2]}(t, t_0), \mathbf{u}_3^{[2]}(t, t_0), \right. \\
& (t - t_0) \mathbf{u}_1^{[2]}(t, t_0), (t - t_0) \mathbf{u}_2^{[2]}(t, t_0), (t - t_0) \mathbf{u}_3^{[2]}(t, t_0), \\
& \left. (t - t_0)^2 \mathbf{u}_1^{[2]}(t, t_0), (t - t_0)^2 \mathbf{u}_2^{[2]}(t, t_0), (t - t_0)^2 \mathbf{u}_3^{[2]}(t, t_0) \right\}
\end{aligned}$$

is linearly independent on  $[t_0, t_f]$ .

*Proof:* Suppose that the set of functions  $\mathcal{F}_r$  is linearly dependent. Then, it is clear that there exists a nonzero vector  $\mathbf{c} = [\mathbf{c}_1^T \mathbf{c}_2^T \mathbf{c}_3^T]^T \in \mathbb{R}^9$  such that

$$\mathbf{c} \cdot \begin{bmatrix} \mathbf{u}^{[2]}(t, t_0) \\ (t - t_0) \mathbf{u}^{[2]}(t, t_0) \\ (t - t_0)^2 \mathbf{u}^{[2]}(t, t_0) \end{bmatrix} = \mathbf{0} \quad (17)$$

for all  $t \in [t_0, t_f]$ . Let  $y^a(t)$  denote the output of (5) with initial condition  $\mathbf{x}_1^a(t_0) = -2\mathbf{c}_1$ ,  $\mathbf{x}_2^a(t_0) = 2\mathbf{c}_2$ , and  $\mathbf{x}_3^a(t_0) = \mathbf{c}_3$  and  $y^b(t)$  denote the output of (5) with initial condition  $\mathbf{x}_1^b(t_0) = 2\mathbf{c}_1$ ,  $\mathbf{x}_2^b(t_0) = -2\mathbf{c}_2$ , and  $\mathbf{x}_3^b(t_0) =$

$-\mathbf{c}_3$ . It is straightforward to show, using (17), that  $y^a(t) = y^b(t)$  for all  $t \in [t_0, t_f]$ . Thus, if the set of functions  $\mathcal{F}_r$  is not linearly independent, there exist, at least, two states that are indistinguishable. Therefore, the system is not observable, which concludes the proof. ■

To conclude this section, it is important to remark that, although all the results were derived concerning the observability of the nonlinear system (5), these also apply to the observability of the original nonlinear system (3) as they are related by the Lyapunov transformation (4).

#### IV. FILTER DESIGN

In the previous section the observability of (3) was discussed by means of state augmentation and a coordinate change and Theorem 3 provides a practical result on the design of a state observer for (5). In order to recover the augmented system dynamics in body-fixed coordinates, consider the state transformation  $\mathbf{x}(t) = \mathbf{T}_c^T(t)\mathbf{x}(t)$ , where

$$\mathbf{T}_c(t) = \text{diag}(\mathbf{T}(t), 1, 1, 1, 1, 1). \quad (18)$$

Notice that (18) is a Lyapunov transformation matrix [11]. The augmented system dynamics in body-fixed coordinates are given by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\mathbf{a}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases},$$

where

$$\mathbf{A}(t) = \begin{bmatrix} -\mathbf{S}(\omega(t)) & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}(\omega(t)) & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{S}(\omega(t)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & -\frac{1}{y(t)} & 0 & 0 & 0 \\ \mathbf{a}^T(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{0} & -2\mathbf{a}^T(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & -3 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{a}^T(t) & \mathbf{0} & \mathbf{0} & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \end{bmatrix}.$$

Adding system disturbances  $\mathbf{n}_x$  and sensor noise  $n_y$ , the design of a Kalman filter follows for the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}\mathbf{a}(t) + \mathbf{n}_x(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + n_y(t) \end{cases},$$

where it is assumed that  $\mathbf{n}_x$  and  $n_y$  are uncorrelated zero-mean Gaussian noise signals, with  $E[\mathbf{n}_x(t)\mathbf{n}_x^T(\tau)] = Q\delta(t - \tau)$  and  $E[n_y(t)n_y(\tau)] = R\delta(t - \tau)$ .

To conclude this section notice that stronger forms of observability should be imposed for the stability of the Kalman filter. In particular, if (19) is uniformly complete observable, then the Kalman filter is stable [14], [15]. This form of observability is closely related to the one previously derived and it is rather straightforward to derive conditions for uniform complete observability based on the previous results.

#### V. SIMULATIONS

In order to evaluate the performance achieved with the proposed navigation solution, simulations were carried out with a full nonlinear model of a quadrotor (see [16] for further details). In the simulations a trajectory tracking control law [16] was employed to follow the desired trajectory

$$\mathbf{p}_d(t) = \begin{bmatrix} 20 + \frac{100}{2\pi} \sin\left(\frac{2\pi}{60}t\right) \\ 20 + \frac{50}{2\pi} \cos\left(\frac{2\pi}{30}t + \frac{\pi}{3}\right) \\ 20 + \frac{20}{2\pi} \sin\left(\frac{2\pi}{40}t - \frac{\pi}{4}\right) \end{bmatrix} (\text{m}).$$

The resulting trajectory is shown in Fig. 1.

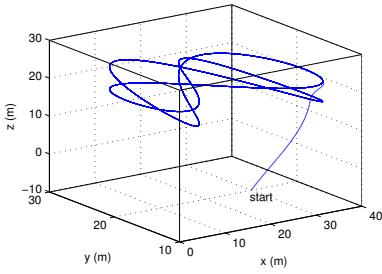


Fig. 1. Trajectory described by the quadrotor

Sensor noise was considered for all sensors. In particular, the range, acceleration, and angular velocity measurements are assumed to be corrupted with additive uncorrelated zero-mean white Gaussian noises, with standard deviations of 0.2 m, 0.001 m/s<sup>2</sup>, and 0.001 °/s, respectively. The source was fixed at  $[1 \ 1 \ 1]^T$  (m). These are realistic measurements considering the scale of the problem.

To tune the Kalman filter the state disturbance covariance matrix was chosen as

$$\mathbf{Q} = 10^{-5} \text{diag}(10\mathbf{I}, 10\mathbf{I}, 0.1\mathbf{I}, 1, 0.2, 0.2, 0.1, 0.01)$$

and the output noise variance as  $R = 1$ . The evolution of the position, velocity, and acceleration of gravity errors is depicted in Fig. 2, whereas the evolution of the remaining state errors is omitted due to the lack of space.

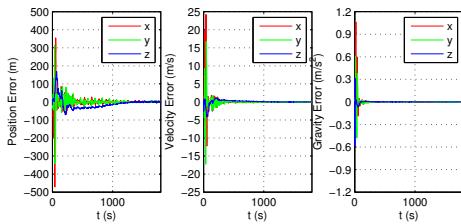


Fig. 2. Time evolution of the position, velocity, and acceleration of gravity error

The initial large transients appear due to the mismatch of the initial conditions and last for a while due to the tremendous estimation effort that is to estimate the position, velocity, and acceleration of gravity from the observation of the norm of a position measurement. In spite of that, good performance is attained and the steady-state error remains confined to a very tight interval. The standard deviation of the position, velocity, and acceleration of gravity error is shown in Table I.

TABLE I

STANDARD DEVIATION OF THE ERROR OF THE POSITION, VELOCITY, AND ACCELERATION OF GRAVITY

Error variable	x-axis	y-axis	z-axis
Position (m)	2.3	2.1	0.43
Velocity (m/s)	0.044	0.038	0.0046
Gravity (m/s <sup>2</sup> )	$0.60 \times 10^{-3}$	$0.57 \times 10^{-3}$	$0.11 \times 10^{-3}$

## VI. CONCLUSION

The problem of navigation of autonomous vehicles based on the range to a single source was addressed in this paper. The observability of the system was analyzed and practical meaningful necessary and sufficient conditions were derived that characterize this aspect of the system. These conditions are closely related to the motion of the vehicle and allow for the implementation of appropriate motion planning and control strategies that render the system observable. To solve the estimation problem a standard Kalman filter is proposed for a linear system that mimics the exact dynamics of the nonlinear range-based system under specific observability conditions. Since no linearizations are considered, the stability of the filter is well characterized from classic Kalman filter theory results. Simulation results were presented that evidence the attainable performance in the presence of realistic sensor noise.

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